John Long’s Hilbert Space

The “vectors” in the original linear space are Zeno contours: contours in the complex plane that are given parametrically as \( z(t) \) or \( \gamma(t) \), \( 0 \leq t \leq 1 \), as solutions of the differential equations

\[
\frac{dz}{dt} = f(z,t) - t,
\]

where \( f(z,t) \) is an underlying time-dependent vector field, or algorithmically as

\[
z_{k,n} = z_{k-1,n} + \frac{1}{n}\left(f(z_{k-1,n}, \frac{k}{n}) - z_{k-1,n}\right)
\]

with \( k: 1 \to n \) and \( n \to \infty \). Assume \( f(z,t) \) is continuous in \( \mathbb{C} \times [0,1] \).

Define an inner product that provides an abstract version of orthogonality (intersection of two lines at right angles):

\[
\langle \gamma_1, \gamma_2 \rangle = \int_0^1 \gamma_1 \overline{\gamma_2} \, dt,
\]

from which one infers a norm of the space (“how far the vector is from the origin”):

\[
\| \gamma \| = \sqrt{\langle \gamma, \gamma \rangle}.
\]

This norm induces a metric for the space (defines “distance between vectors”):

\[
d(\gamma_1, \gamma_2) = \| \gamma_1 - \gamma_2 \|.
\]

Linear operators are employed on Hilbert Spaces. A very simple example is:

\[
O(\gamma) = e^{i \pi/4} \gamma.
\]

Comment: There is nothing original here . . . simply an elementary example of a Hilbert Space for an aging rock climber and writer who loves them. Physicists use Hilbert Spaces as computational tools and employ somewhat different notations.